XXVIII. "On Hamilton's Numbers." By J. J. SYLVESTER, F.R.S., Savilian Professor of Geometry in the University of Oxford, and JAMES HAMMOND, M.A. Cant. Received June 11, 1887.

(Abstract.)

In the year 1786 Erland Samuel Bring, Professor at the University of Lund in Sweden, discovered that by the method of Tschirnhausen it was possible to deprive the general algebraical equation of the 5th degree of three of its terms without solving an equation higher than the 3rd degree. By a well understood, however singular, academical fiction, this discovery was imputed by him to one of his own pupils, one Sven Gustaf Sommelius, and embodied in a thesis humbly submitted to himself for approval by that pupil, as a preliminary to his obtaining his degree of Doctor of Philosophy in the University.* It seems to have been overlooked or forgotten, and was subsequently re-discovered many years later by Mr. Jerrard. In a report contained in the 'Proceedings of the British Association' for 1836, Sir William Hamilton showed that Mr. Jerrard was mistaken in supposing that the method was adequate to taking away more than three terms of the equation of the 5th degree, but supplemented this somewhat unnecessary refutation by a profound and original discussion of a question raised by Mr. Jerrard, as to the number of variables required in order that any system of equations of given degrees in those variables shall admit of being satisfied without solving any equation of a degree higher than the highest of the given degrees.

In the year 1886 the senior author of this memoir showed in a paper in Kronecker's (better known as Crelle's) 'Journal' that the trinomial equation of the 5th degree, upon which by Bring's method the general equation of that degree can be made to depend, has necessarily imaginary coefficients except in the case where four of the roots of the original equation are imaginary, and also pointed out a method of obtaining the absolute minimum degree M of an equation from which any given number of specified terms can be taken away subject to the condition of not having to solve any equation of a

* Bring's Reduction of the Quintic Equation was republished by Mr. Robert Harley, F.R.S., in the 'Quarterly Journal of Pure and Applied Mathematics,' vol. 6, 1864, p. 45. The full title of the Lund Thesis, as given by Mr. Harley (see 'Quart. Journ. Math.,' pp. 44, 45) is as follows: "B. cum D. Meletemata quaedam mathematica circa transformationem aequationum algebraicarum, quae consent. Ampliss. Facult. Philos. in Regia Academia Carolina Praeside D. Erland Sam. Bring, Hist. Profess. Reg. & Ord. publico Eruditorum Examini modeste subjicit Sven Gustaf Sommelius, Stipendiarius Regius & Palmcrentzianus Lundensis. Die XIV Decemb., MDCCLXXXVI, L.H.Q.S.—Lundae, typis Berlingianis."

degree higher than M. The numbers furnished by Hamilton's method, it is to be observed, are not *minima* unless a more stringent condition than this is substituted, viz., that the system of equations which have to be resolved in order to take away the proposed terms shall be the simplest possible, *i.e.*, of the lowest possible weight and not merely of the lowest order; in the memoir in 'Crelle' above referred to, he has explained in what sense the words weight and order are here employed. He has given the name of Hamilton's Numbers to these relative minima (minima, *i.e.*, in regard to weight), for the case where the terms to be taken away from the equation occupy consecutive places in it, beginning with the second.

Mr. James Hammond has quite recently discovered by the method of generating functions a very simple formula of reduction, or scale of relation, whereby any one of these numbers may be expressed in terms of those that precede it: his investigation, which constitutes its most valuable portion, will be found in the second section of this paper. The principal results obtained by its senior author consequential in great measure to Mr. Hammond's remarkable and unexpected discovery, refer to the proof of a theorem left undemonstrated in the memoir in 'Crelle' above referred to, and the establishment of certain other asymptotic laws to which Hamilton's Numbers and their differences are subject, by a mixed kind of reasoning, in the main apodictic, but in part also founded on observation. It thus became necessary to calculate out the 10th Hamiltonian Number, which contains 43 places of figures. The highest number calculated by Hamilton (the 6th) was the number 923, which comes third in order after 5 (the Bring number), 11 and 47 being the two intervening numbers. It is to be hoped that some one will be found willing to undertake the labour (considerable but not overwhelming) of calculating some further numbers in the scale, in order to establish or disprove conclusively the presumptive law of the asymptotic branch of the series connecting any two consecutive semi-differences η_x, η_{x+1} of the Hamiltonian Numbers, viz.:—

$$\eta_{x+1} - \eta_x^2 = \eta_x^{\frac{1}{2}} \sum_{r=0}^{r=\infty} c_r \, \eta_x^{(\frac{1}{2})^r}$$

The theory has been "a plant of slow growth." The Lund Thesis of December, 1786 (a matter of a couple of pages), Hamilton's Report of 1836, with the tract of Mr. Jerrard therein referred to, and the memoir in 'Crelle' of December, 1886, constitute as far as the senior author of this paper is aware, the complete bibliography of the subject up to the present date.